

# 9

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## Multi-period Planning Problems

### 9.1 Introduction

One of the most important uses of optimization is in multi-period planning. Most of the problems we have considered thus far have been essentially one-period problems. The formulations acted as if decisions this period were decoupled from decisions in future periods. Typically, however, if we produce more of a certain product this period than required by a constraint, that extra production will not be worthless, but can probably be used next period.

These interactions between periods can be represented very easily within optimization models. In fact, most large linear programs encountered in practice are multi-period models. A common synonym for “multi-period” is “dynamic” (e.g., a multi-period LP may be referred to as a dynamic model).

In some applications, the need to represent the multi-period aspects is quite obvious. One setting in which multi-period LP has been used for a number of years is in the manufacture of cheese. Production decisions must be made monthly or even weekly. The production time for many cheeses, however, may be months. For example, Parmesan cheese may need to be stored in inventory for up to ten months. What Americans call Swiss cheese may take from two to four months. The various grades of cheddar obtained depend upon the number of weeks held in storage. Sharp cheddar may be aged up to a year in storage. Clearly, in such applications, the multi-period aspect of the model is the important feature.

Models for planning over time represent the real world by partitioning time into a number of periods. The portion of the model corresponding to a single period might be some combination of product mix, blending, and other models. These single-period or static models are linked by:

1. A link or inventory variable for each commodity and period. The linking variable represents the amount of commodity transferred from one period to the next.
2. A “material balance” or “sources = uses” constraint for each commodity and period. The simplest form of this constraint is “beginning inventory + production = ending inventory + goods sold”.

Multi-period models are usually used in a rolling or sliding format. In this format, the model is solved at the beginning of each period. The recommendations of the solution for the first period are implemented. As one period elapses and better data and forecasts become available, the model is slid forward one period. The period that had been number 2 becomes number 1, etc., and the whole process

is repeated. When using a model in this sliding fashion, a practical problem is that, as the new information becomes available, this period's "optimal" solution may be drastically different from the previous period's "optimal" solution. The people who have to implement the solution may find this disconcerting. The scheduling system is said to suffer from nervousness. An approach that has been used successfully in scheduling ships, scheduling plant closings/openings, and scheduling production of breakfast cereal, see Brown, Dell, and Wood (1997), is to specify a "reference" solution (e.g., the previous period's solution). One then defines a secondary objective of minimizing the deviation of the current solution from the reference solution. If one puts zero weight on the secondary objective, then one gets the theoretically optimal solution. If one puts an extremely high weight on the secondary objective, then one simply gets the reference solution returned. If a modest weight is placed on the secondary objective, then one gets a solution that is a good compromise between low cost as measured by standard accounting, but is also close to the reference solution.

There is nothing sacred about having all periods of the same length. For example, when a petroleum company plans production for the coming year, it is sensible to have the periods correspond to the seasons of the year. One possible partition is to have the winter period extend from December 1 to March 15, the spring period extend from March 16 to May 15, the summer period extend from May 16 to September 15, and the autumn period extend from September 16 to November 30.

Some companies, such as forest product or mineral resource based companies, plan as much as 50 years into the future. In such a case, one might have the first two periods be one year each, the next period be two years, the next two periods three years each, the next two periods five years each, and the final three periods ten years each.

Inter-period interactions are usually accounted for in models by the introduction of inventory decision variables. These variables "link" adjacent periods. As an example, suppose we have a single explicit decision to make each period. Namely, how much to produce of a single product. Call this decision variable for period  $j$ ,  $P_j$ . Further, suppose we have contracts to sell known amounts of this product,  $d_j$ , in period  $j$ . Define the decision variable  $I_j$  as the amount of inventory left over at the end of period  $j$ . By this convention, the beginning inventory in period  $j$  is  $I_{j-1}$ . The LP formulation will then contain one "sources of product = uses of product" constraint for each period. For period 2, the sources of product are beginning inventory,  $I_1$ , and production in the period,  $P_2$ . The uses of product are demand,  $d_2$ , and end of period inventory,  $I_2$ . For example, if  $d_2 = 60$  and  $d_3 = 40$ , then the constraint for period 2 is:

$$I_1 + P_2 = 60 + I_2 \quad \text{or} \quad I_1 + P_2 - I_2 = 60.$$

The constraint for period 3 is:

$$I_2 + P_3 - I_3 = 40.$$

Notice how  $I_2$  "links" (i.e., appears in both the constraints for periods 2 and 3).

In some problems, the net outflow need not exactly equal the net inflow into the next period. For example, if the product is cash, then one of the linking variables may be short-term borrowing or lending. For each dollar carried over from period 2 by lending, we will enter period 3 with \$1.05 if the interest rate is 5% per period.

On the other hand, if the "product" is workforce and there is a predictable attrition rate of 10% per period, then the above two constraints would be modified to:

$$\begin{aligned} .90I_1 + P_2 - I_2 &= 60 \\ .90I_2 + P_3 - I_3 &= 40. \end{aligned}$$

In this case,  $P_i$  is the number hired in period  $i$ .

The following example provides a simplified illustration of a single-product, multi-period planning situation.

## 9.2 A Dynamic Production Problem

A company produces one product for which the demand for the next four quarters is predicted to be:

Spring	Summer	Autumn	Winter
20	30	50	60

Assuming all the demand is to be met, there are two extreme policies that might be followed:

1. “Track” demand with production and carry no inventory.
2. Produce at a constant rate of 40 units per quarter and allow inventory to absorb the fluctuations in demand.

There are costs associated with carrying inventory and costs associated with varying the production level, so one would expect the least-cost policy is probably a combination of (1) and (2) (i.e., carry some inventory, but also vary the production level somewhat).

For costing purposes, the company estimates changing the production level from one period to the next costs \$600 per unit. These costs are often called “hiring and firing” costs. It is estimated that charging \$700 for each unit of inventory at the end of the period can accurately approximate inventory costs. The initial inventory is zero and the initial production level is 40 units per quarter. We require these same levels be achieved or returned to at the end of the winter quarter.

We can now calculate the production change costs associated with the no-inventory policy as:

$$\$600 \times (20 + 10 + 20 + 10 + 20) = \$48,000.$$

On the other hand, the inventory costs associated with the constant production policy is:

$$\$700 \times (20 + 30 + 20 + 0) = \$49,000.$$

The least cost policy is probably a mix of these two pure policies. We can find the least-cost policy by formulating a linear program.

### 9.2.1 Formulation

The following definitions of variables will be useful:

$P_i$  = number of units produced in period  $i$ , for  $i = 1, 2, 3$ , and  $4$ ;

$I_i$  = units in inventory at the end of period  $i$ ;

$U_i$  = increase in production level between period  $i - 1$  and  $i$ ;

$D_i$  = decrease in production level between  $i - 1$  and  $i$ .

The  $P_i$  variables are the obvious decision variables. It is useful to define the  $I_i$ ,  $U_i$ , and  $D_i$  variables, so we can conveniently compute the costs each period.

To minimize the cost per year, we want to minimize the sum of inventory costs:

$$\$700 I_1 + \$700 I_2 + \$700 I_3 + \$700 I_4$$

plus production change costs:

$$\begin{aligned} & \$600 U_1 + \$600 U_2 + \$600 U_3 + \$600 U_4 + \$600 U_5 \\ & + \$600 D_1 + \$600 D_2 + \$600 D_3 + \$600 D_4 + \$600 D_5. \end{aligned}$$

We have added a  $U_5$  and a  $D_5$  in order to charge for the production level change back to 40, if needed at the end of the 4<sup>th</sup> period.

## 9.2.2 Constraints

Every multi-period problem will have a “material balance” or “sources = uses” constraint for each product per period. The usual form of these constraints in words is:

$$\textit{beginning inventory} + \textit{production} - \textit{ending inventory} = \textit{demand}.$$

Algebraically, these constraints for the problem at hand are:

$$\begin{aligned} P_1 - I_1 &= 20 \\ I_1 + P_2 - I_2 &= 30 \\ I_2 + P_3 - I_3 &= 50 \\ I_3 + P_4 &= 60 \end{aligned}$$

Notice  $I_4$  and  $I_0$  do not appear in the first and last constraints, because initial and ending inventories are required to be zero.

If the formulation is solved as is, there is nothing to force  $U_1$ ,  $D_1$ , etc., to be greater than zero. Therefore, the solution will be the pure production policy. Namely,  $P_1 = 20$ ,  $P_2 = 30$ ,  $P_3 = 50$ ,  $P_4 = 60$ . This policy implies a production increase at the end of every period, except the last. This suggests a way of forcing  $U_1$ ,  $U_2$ ,  $U_3$ , and  $U_4$  to take the proper values is to append the constraints:

$$\begin{aligned} U_1 &\geq P_1 - 40 \\ U_2 &\geq P_2 - P_1 \\ U_3 &\geq P_3 - P_2 \\ U_4 &\geq P_4 - P_3. \end{aligned}$$

Production decreases are still not properly measured. An analogous set of four constraints should take care of this problem, specifically:

$$\begin{aligned} D_1 &\geq 40 - P_1 \\ D_2 &\geq P_1 - P_2 \\ D_3 &\geq P_2 - P_3 \\ D_4 &\geq P_3 - P_4. \end{aligned}$$

To incorporate the requirement that the production level be returned to 40 at the end of the winter quarter, we add the variables  $U_5$  and  $D_5$  to measure changes at the end of the last quarter.  $U_5$  and  $D_5$  are forced to take on the right values with the constraints:

$$\begin{aligned} U_5 &\geq 40 - P_4 \\ D_5 &\geq P_4 - 40. \end{aligned}$$

Before moving on, we will note the production-change constraints can be reduced to 5 constraints from the 10 implied by the above form. The key observation is two constraints such as:

$$\begin{aligned} U_2 &\geq P_2 - P_1 \\ D_2 &\geq P_1 - P_2 \end{aligned}$$

can be replaced by the single constraint:

$$U_2 - D_2 = P_2 - P_1.$$

The argument is more economic than algebraic. The purpose with either formulation is to force  $U_2 = P_2 - P_1$  if  $P_2 - P_1 \geq 0$  and  $D_2 = P_1 - P_2$  if  $P_1 - P_2 \geq 0$ . From economics, you can argue that, at the optimal solution, you will find at most one of  $U_2$  and  $D_2$  are greater than 0 under either formulation. If both  $U_2$  and  $D_2$  are greater than 0 under the second formulation, then both can be reduced by an equal amount. Thus, reducing costs without violating any constraints.

The complete formulation is:

```

MODEL:
!Minimize inventory + workforce change costs;
MIN = 700 * I1 + 700 * I2 + 700 * I3 + 700 * I4
      + 600 * U1 + 600 * U2 + 600 * U3 + 600 * U4
      + 600 * D1 + 600 * D2 + 600 * D3 + 600 * D4
      + 600 * U5 + 600 * D5;
!Initial conditions on inventory & production;
[CNDBI] I0 = 0;
[CNDBP] P0 = 40;
!Beginning inventory + production = demand + ending inventory;
[INV1] I0 + P1 = 20 + I1;
[INV2] I1 + P2 = 30 + I2;
[INV3] I2 + P3 = 50 + I3;
[INV4] I3 + P4 = 60 + I4;
!Change up - change down = prod. this period - prod. prev. period;
[CHG1] U1 - D1 = P1 - P0;
[CHG2] U2 - D2 = P2 - P1;
[CHG3] U3 - D3 = P3 - P2;
[CHG4] U4 - D4 = P4 - P3;
[CHG5] U5 - D5 = P5 - P4;
!Ending conditions;
[CNDEI] I4 = 0;
[CNDEP] P5 = 40;
END

```

The solution is:

Optimal solution found at step:		7
Objective value:		43000.00
Variable	Value	Reduced Cost
I1	5.000000	0.000000
I2	0.000000	200.0000
I3	5.000000	0.000000
I4	0.000000	0.000000
U1	0.000000	1200.000
U2	0.000000	250.0000
U3	30.00000	0.000000
U4	0.000000	250.0000
D1	15.00000	0.000000
D2	0.000000	950.0000
D3	0.000000	1200.000
D4	0.000000	950.0000
U5	0.000000	1200.000

D5	15.00000	0.0000000
I0	0.0000000	0.0000000
P0	40.00000	0.0000000
P1	25.00000	0.0000000
P2	25.00000	0.0000000
P3	55.00000	0.0000000
P4	55.00000	0.0000000
P5	40.00000	0.0000000
Row	Slack or Surplus	Dual Price
1	43000.00	-1.000000
CNDBI	0.0000000	-950.0000
CNDBP	0.0000000	-600.0000
INV1	0.0000000	950.0000
INV2	0.0000000	250.0000
INV3	0.0000000	-250.0000
INV4	0.0000000	-950.0000
CHG1	0.0000000	600.0000
CHG2	0.0000000	-350.0000
CHG3	0.0000000	-600.0000
CHG4	0.0000000	-350.0000
CHG5	0.0000000	600.0000
CNDEI	0.0000000	-1650.000
CNDEP	0.0000000	600.0000

We see the solution is a mixed policy:

$$P_1 = P_2 = 25; \quad P_3 = P_4 = 55.$$

The mixed policy found by LP is \$5,000 cheaper than the best pure policy.

### 9.2.3 Representing Absolute Values

You may be tempted to represent the production-change costs in the above model by the expression:

$$600 * ( @ABS ( P1 - P0 ) + @ABS ( P2 - P1 ) + ... + @ABS ( P5 - P4 ) );$$

This is mathematically correct, but computationally unwise, because it converts a linear program into a nonlinear program. Nonlinear programs are always more time consuming to solve. We have exploited the following result to obtain a linear program from an apparently nonlinear program. Subject to a certain condition, any appearance in a model of a term of the form:

$$@ABS ( expression )$$

can be replaced by the term  $U + D$ , if we add the constraint:

$$U - D = expression.$$

The “certain condition” is that the model must be such that a small value of  $@ABS ( expression )$  is preferred to a large value for  $@ABS ( expression )$ . The result is, if  $expression$  is positive, then  $U$  will be equal to  $expression$ , whereas, if  $expression$  is negative, then  $D$  will equal the negative of  $expression$ .

## 9.3 Multi-period Financial Models

In most multi-period planning problems, the management of liquid or cash-like assets is an important consideration. If you are willing to consider cash holdings as an inventory just like an inventory of any

other commodity, then it is a small step to incorporate financial management decisions into a multi-period model. The key feature is, for every period, there is a constraint that effectively says, “sources of cash – uses of cash = 0”. The following simple, but realistic, example illustrates the major features of such models.

### 9.3.1 Example: Cash Flow Matching

Suppose, as a result of a careful planning exercise, you have concluded that you will need the following amounts of cash for the current plus next 14 years to meet certain commitments:

<b>Year:</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>	<b>13</b>	<b>14</b>
<b>Cash (in \$1,000s)</b>	10	11	12	14	15	17	19	20	22	24	26	29	31	33	36

A common example where such a projection is made is in a personal injury lawsuit. Both parties may reach an agreement that the injured party should receive a stream of payments such as above or its equivalent. Other examples where the above approach has been used is in designing bond portfolios to satisfy cash needs for a pension fund, or for so-called balance sheet defeasance where one kind of debt is replaced by another having the same cash flow stream.

For administrative simplicity in the personal injury example, both parties prefer an immediate single lump sum payment that is “equivalent” to the above stream of 15 payments. The party receiving the lump sum will argue that the lump sum payment should equal the present value of the stream using a low interest rate such as that obtained in a very low risk investment (i.e., a government guaranteed savings account). For example, if an interest rate of 4% is used, the present value of the stream of payments is \$230,437. The party that must pay the lump sum, however, would like to argue for a much higher interest rate. To be successful, such an argument must include evidence that such higher interest rate investments are available and are no riskier than savings accounts. The investments usually offered are government securities. Generally, a broad spectrum of such investments is available on a given day. For simplicity, assume there are just two such investments available with the following features:

<b>Security</b>	<b>Current Cost</b>	<b>Yearly Return</b>	<b>Years to Maturity</b>	<b>Principal Repayment at Maturity</b>
1	\$980	\$60	5	\$1000
2	\$965	\$65	12	\$1000

The paying party will offer a lump sum now with a recommendation of how much should be invested in securities 1 and 2 and in savings accounts, such that the yearly cash requirements are met with the minimum lump sum payment.

The following decision variables are useful in solving this problem:

- $B_1$  = amount invested now in security 1, measured in “face value amount”,
- $B_2$  = amount invested now in security 2, measured in “face value amount”,
- $S_i$  = amount invested into a savings account in year  $i$ , and
- $L$  = initial lump sum.

The objective function will be to minimize the initial lump sum. There will be a constraint for each year that forces the cash flows to net to zero. If we assume idle cash is invested at 4 percent in a savings account and all amounts are measured in \$1000's, then the formulation is:

$$\begin{aligned}
 \text{MIN} &= L; \\
 L - 0.98 * B1 - 0.965 * B2 - S0 &= 10; \\
 0.06 * B1 + 0.065 * B2 + 1.04 * S0 - S1 &= 11; \\
 0.06 * B1 + 0.065 * B2 + 1.04 * S1 - S2 &= 12; \\
 0.06 * B1 + 0.065 * B2 + 1.04 * S2 - S3 &= 14; \\
 0.06 * B1 + 0.065 * B2 + 1.04 * S3 - S4 &= 15; \\
 1.06 * B1 + 0.065 * B2 + 1.04 * S4 - S5 &= 17; \\
 0.065 * B2 + 1.04 * S5 - S6 &= 19; \\
 0.065 * B2 + 1.04 * S6 - S7 &= 20; \\
 0.065 * B2 + 1.04 * S7 - S8 &= 22; \\
 0.065 * B2 + 1.04 * S8 - S9 &= 24; \\
 0.065 * B2 + 1.04 * S9 - S10 &= 26; \\
 0.065 * B2 + 1.04 * S10 - S11 &= 29; \\
 1.065 * B2 + 1.04 * S11 - S12 &= 31; \\
 1.04 * S12 - S13 &= 33; \\
 1.04 * S13 - S14 &= 36;
 \end{aligned}$$

The PICTURE of the constraint coefficients gives a better appreciation of the structure of the problem. An *A* represents numbers bigger than 1.0, but less than 10.0. Numbers 10 or larger, but less than 100.0, are represented by a *B*. Numbers less than 1.0, but at least 0.1, are represented by a *T*. Numbers less than 0.1, but at least 0.01, are represented by a *U*:

	B	B	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S
L	1	2	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4					
1:	1																					
2:	1-T	T	T	-1																		MIN
3:	U	U		A-1																		= B
4:	U	U			A-1																	= B
5:	U	U				A-1																= B
6:	U	U					A-1															= B
7:	A	U						A-1														= B
8:	U								A-1													= B
9:	U									A-1												= B
10:	U										A-1											= B
11:	U											A-1										= B
12:	U												A-1									= B
13:	U													A-1								= B
14:	A														A-1							= B
15:																A-1						= B
16:																	A-1					= B

Notice in row 7,  $B_1$  has a coefficient of 1.06. This represents the principal repayment of \$1000 plus the interest payment of \$60 measured in \$1000's. Variable  $S_{14}$  (investment of funds in a savings account after the final payment is made) appears in the problem even though at first you might think it useless to allow such an option.  $S_{14}$  is effectively a surplus cash variable in the final period. Nevertheless, it is not unusual for the solution that minimizes the lump sum payment to have cash left over at the end of the period. This is because a bond may be the most economical way of delivering funds to intermediate periods. This may cause the big principal repayment at the end of a bond's life to "overpay" the most distant periods. The solution is:

Optimal solution found at step:		14
Objective value:		195.6837
Variable	Value	Reduced Cost
L	195.6837	0.0000000
B1	95.79577	0.0000000
B2	90.15474	0.0000000
S0	4.804497	0.0000000
S1	5.604481	0.0000000
S2	5.436464	0.0000000
S3	3.261727	0.0000000
S4	0.0000000	0.1069792
S5	90.40358	0.0000000
S6	80.87978	0.0000000
S7	69.97503	0.0000000
S8	56.63409	0.0000000
S9	40.75951	0.0000000
S10	22.24994	0.0000000
S11	0.0000000	0.1412458
S12	65.01479	0.0000000
S13	34.61538	0.0000000
S14	0.0000000	0.3796368

Of the \$195,683.70 lump sum payment, \$10,000 goes to immediate requirements, \$4,804.50 goes into a savings account, and  $0.98 \times 95,795.77 + 0.965 \times 90,154.74 = \$180,879.20$  goes into longer-term securities. Considering a wide range of investments rather than just savings accounts has reduced the amount of the lump sum payment by about \$34,750, or 15%.

In actual solutions, one may find a major fraction of the lump sum is invested in a single security. For example, appending the following constraint limits the amount invested in security 1 to half the initial lump sum:

$$0.98 B_1 - 0.5 L \leq 0.$$

An additional complication may arise due to integrality requirements on the  $B_1$  and  $B_2$  investments. For example, bonds can be bought only in \$1000 increments. Generally, with a modest amount of judgment, the fractional values can be rounded to neighboring integer values with no great increase in lump sum payment. For example, if  $B_1$  and  $B_2$  are set to 96 and 90 in the previous example, the total cost increases to \$195,726.50 from \$195,683.70. When this is done,  $S_{14}$  becomes nonzero. Specifically, the last period is overpaid by about \$40.

A sets version that places an integrality requirement on the bond purchase variables is:

```

MODEL:
! Name= PBOND, Bond portfolio/ cash matching problem: Given cash needs
in each future period, what collection of bonds should we buy to cover
these needs?;
SETS:
BOND/1..2/ : MATAT, ! Matures at period;
              PRICE, ! Purchase price;
              CAMNT, ! Coupon payout each period;
              BUY; ! Amount to buy of each bond;
PERIOD/1..15/:
  NEED, ! Cash needed each period;
SINVEST; ! Short term investment each period;
ENDSETS
DATA:
STRTE = .04; ! Short term interest rate;
MATAT = 6, 13; ! Years to maturity;
PRICE = .980, .965; ! Purchase price in thousands;
CAMNT = .060, .065; ! Coupon amount in thousands;
NEED = 10, 11, 12, 14, 15, 17, 19, 20, 22, 24,
       26, 29, 31, 33, 36; ! Cash needed in
       thousands;
ENDDATA
!-----;
MIN = LUMP;
! First period is slightly special;
LUMP =
NEED(1) + SINVEST( 1) + @SUM( BOND: PRICE * BUY);
! For subsequent periods;
@FOR( PERIOD( I) | I #GT# 1:
  @SUM( BOND( J) | MATAT( J) #GE# I:
    CAMNT( J) * BUY( J) ) +
  @SUM( BOND( J) | MATAT( J) #EQ# I: BUY( J) ) +
  ( 1 + STRTE) * SINVEST( I - 1) =
  NEED( I) + SINVEST( I);
);
! Can only buy integer bonds;
@FOR( BOND( J): @GIN( BUY( J)));
END

```

```

Optimal solution found at step:      28
Objective value:                    195.7265
Branch count:                        3

```

Variable	Value	Reduced Cost
STRTE	0.4000000E-01	0.0000000
LUMP	195.7265	0.0000000
MATAT( 1)	6.000000	0.0000000
MATAT( 2)	13.00000	0.0000000
PRICE( 1)	0.9800000	0.0000000
PRICE( 2)	0.9650000	0.0000000
CAMNT( 1)	0.6000000E-01	0.0000000
CAMNT( 2)	0.6500000E-01	0.0000000
BUY( 1)	96.00000	0.7622063

BUY ( 2)	90.00000	0.7290568
NEED ( 1)	10.00000	0.0000000
NEED ( 2)	11.00000	0.0000000
NEED ( 3)	12.00000	0.0000000
NEED ( 4)	14.00000	0.0000000
NEED ( 5)	15.00000	0.0000000
NEED ( 6)	17.00000	0.0000000
NEED ( 7)	19.00000	0.0000000
NEED ( 8)	20.00000	0.0000000
NEED ( 9)	22.00000	0.0000000
NEED ( 10)	24.00000	0.0000000
NEED ( 11)	26.00000	0.0000000
NEED ( 12)	29.00000	0.0000000
NEED ( 13)	31.00000	0.0000000
NEED ( 14)	33.00000	0.0000000
NEED ( 15)	36.00000	0.0000000
SINVEST ( 1)	4.796526	0.0000000
SINVEST ( 2)	5.598387	0.0000000
SINVEST ( 3)	5.432322	0.0000000
SINVEST ( 4)	3.259615	0.0000000
SINVEST ( 5)	0.0000000	0.8548042
SINVEST ( 6)	90.61000	0.0000000
SINVEST ( 7)	81.08440	0.0000000
SINVEST ( 8)	70.17778	0.0000000
SINVEST ( 9)	56.83489	0.0000000
SINVEST ( 10)	40.95828	0.0000000
SINVEST ( 11)	22.44661	0.0000000
SINVEST ( 12)	0.1944784	0.0000000
SINVEST ( 13)	65.05226	0.0000000
SINVEST ( 14)	34.65435	0.0000000
SINVEST ( 15)	0.4052172E-01	0.0000000

## 9.4 Financial Planning Models with Tax Considerations

The next example treats a slightly more complicated version of the portfolio selection problem and then illustrates how to include and examine the effect of taxes. Winston-Salem Development Management (WSDM) is trying to complete its investment plans for the next three years. Currently, WSDM has two million dollars available for investment. At six-month intervals over the next three years, WSDM expects the following income stream from previous investments: \$500,000 (six months from now); \$400,000; \$380,000; \$360,000; \$340,000; and \$300,000 (at the end of third year). There are three development projects in which WSDM is considering participating. The Foster City Development would, if WSDM participated fully, have the following cash flow stream (projected) at six-month intervals over the next three years (negative numbers represent investments, positive numbers represent income): -\$3,000,000; -\$1,000,000; -\$1,800,000; \$400,000; \$1,800,000; \$1,800,000; \$5,500,000. The last figure is its estimated value at the end of three years. A second project involves taking over the operation of some old lower-middle-income housing on the condition that certain initial repairs to it be made and that it be demolished at the end of three years. The cash flow stream for this project, if participated in fully, would be: -\$2,000,000; -\$500,000; \$1,500,000; \$1,500,000; \$1,500,000; \$200,000; -\$1,000,000.

The third project, the Disney-Universe Hotel, would have the following cash flow stream (six-month intervals) if WSDM participated fully. Again, the last figure is the estimated value at the end of the three years: -\$2,000,000; -\$2,000,000; -\$1,800,000; \$1,000,000; \$1,000,000; \$1,000,000; \$6,000,000.

WSDM can borrow money for half-year intervals at 3.5 percent interest per half year. At most, 2 million dollars can be borrowed at one time (i.e., the total outstanding principal can never exceed 2 million). WSDM can invest surplus funds at 3 percent per half year.

Initially, we will disregard taxes. We will formulate the problem of maximizing WSDM's net worth at the end of three years as a linear program. If WSDM participates in a project at less than 100 percent, all the cash flows of that project are reduced proportionately.

### 9.4.1 Formulation and Solution of the WSDM Problem

Define:

- $F$  = fractional participation in the Foster City problem;
- $M$  = fractional participation in Lower-Middle;
- $D$  = participation in Disney;
- $B_i$  = amount borrowed in period  $i$  in 1000's of dollars,  $i = 1, \dots, 6$ ;
- $L_i$  = amount lent in period  $i$  in 1000's of dollars,  $i = 1, \dots, 6$ ;
- $Z$  = net worth after the six periods in 1000's of dollars.

The problem formally is then (all numbers will be measured in units of 1000):

```

MODEL:
MAX = Z; ! Max worth at end of final period;
! Uses - sources = supply of cash in each period;
3000 * F + 2000 * M + 2000 * D - B1 + L1 = 2000;
1000 * F + 500 * M + 2000 * D + 1.035 * B1 - 1.03 * L1 - B2 + L2=500;
1800 * F - 1500 * M + 1800 * D + 1.035 * B2 - 1.03 * L2 - B3 + L3=400;
-400 * F - 1500 * M - 1000 * D + 1.035 * B3 - 1.03 * L3 - B4 + L4=380;
-1800 * F - 1500 * M - 1000 * D + 1.035 * B4 - 1.03 * L4 - B5 + L5=360;
-1800 * F - 200 * M - 1000 * D + 1.035 * B5 - 1.03 * L5 - B6 + L6=340;
Z - 5500 * F + 1000 * M - 6000 * D + 1.035 * B6 - 1.03 * L6=300;
! Borrowing limits;
B1 <= 2000;
B2 <= 2000;
B3 <= 2000;
B4 <= 2000;
B5 <= 2000;
B6 <= 2000;
! We can invest at most 100% in a project;
F <= 1;
M <= 1;
D <= 1;
END

```

Rows 4 through 17 are the cash flow constraints for each of the periods. They enforce the requirement that *uses of cash* - *sources of cash* = 0 for each period. In the initial period, for example,  $L_1$  uses cash, whereas  $B_1$  is a source of cash.

The solution is:

Optimal solution found at step:	11	
Objective value:	7665.179	
Variable	Value	Reduced Cost
Z	7665.179	0.0000000
F	0.7143414	0.0000000
M	0.6372096	0.0000000
D	0.0000000	452.3816
B1	1417.443	0.0000000
L1	0.0000000	0.8788487E-02
B2	2000.000	0.0000000
L2	0.0000000	0.3343139
B3	2000.000	0.0000000
L3	0.0000000	0.2509563
B4	448.4490	0.0000000
L4	0.0000000	0.5304549E-02
B5	0.0000000	0.5149997E-02
L5	2137.484	0.0000000
B6	0.0000000	0.5000029E-02
L6	3954.865	0.0000000
Row	Slack or Surplus	Dual Price
1	7665.179	1.000000
2	0.0000000	1.819220
3	0.0000000	1.757701
4	0.0000000	1.381929
5	0.0000000	1.098032
6	0.0000000	1.060900
7	0.0000000	1.030000
8	0.0000000	1.000000
9	582.5567	0.0000000
10	0.0000000	0.3274043
11	0.0000000	0.2454662
12	1551.551	0.0000000
13	2000.000	0.0000000
14	2000.000	0.0000000
15	0.2856586	0.0000000
16	0.3627904	0.0000000
17	1.000000	0.0000000

Thus, we should try to invest or buy 0.7143414 of the Foster City project, 0.6372096 of the Middle-income project, and invest nothing in the Disney Universe project. At the end of the planning horizon, our net worth should have grown to 7,665,179.

### 9.4.2 Interpretation of the Dual Prices

The dual price on each of the first seven constraints is the increase in net worth in the last period resulting from an extra dollar made available in the earliest period. For example, the 1.81922 indicates an extra dollar available at the start of period 1 would increase the net worth in the last period by about \$1.82.

An extra dollar in period 5 is worth \$1.0609 at the end, because all we will do with it is invest it for two periods at three percent. Thus, it will grow to  $1.03 \times 1.03 = 1.0609$  at the end.

An extra dollar in period 4 will save us from borrowing a dollar that period. Thus, we will be \$1.035 richer in period 5. We have already seen the value per extra dollar in period 5, so the value of an extra dollar in period 4 is  $\$1.035 \times 1.0609 = \$1.09803$ .

The dual prices on the borrowing constraints can be reconciled with the rest of the dual prices as follows. Having an additional dollar in period 2 is worth \$1.7577. If this dollar were borrowed, then we would have to pay out \$1.035 in period 3, which would have an effective cost of  $1.035 \times 1.38193$ . Thus, the net value in the last period of borrowing an extra dollar in period 2 is  $1.7577 - 1.035 \times 1.38193 = 0.3274$ , which agrees with the dual price on the borrowing constraint for period 2.

The effective interest rate or cost of capital,  $i$ , in any period  $t$ , can be found from the dual prices by deriving the rate at which one would be willing to borrow. Borrowing one dollar in period  $t$  would give us \$1 more in period  $t$ , but would require us to pay out  $1 + i$  dollars in period  $t + 1$ . We must balance these two considerations. Consider period 1. An extra dollar is worth \$1.81922 at the end of period 6. Paying back  $1 + i$  in period 2 would cost  $(1 + i)$  \$1.7577 at the end of period 6. Balancing these two:

$$1.81922 = (1 + i)1.7577.$$

Solving:

$$i = 0.035.$$

This is not surprising because we are already borrowing at that rate in period 1, but not to the limit. Applying a similar analysis to the other periods, we get the following effective rates:

Period	$i$	Period	$i$
1	0.03500	4	0.035
2	0.27190	5	0.030
3	0.25855	6	0.030

## 9.5 Present Value vs. LP Analysis

A standard method for evaluating the attractiveness of a project is by computing the present value of its cash flow stream. LP analysis, as we have just illustrated, is a generalization of present value (PV) analysis. The assumptions underlying PV analysis are that money can be: a) borrowed or lent at the same rate, b) without limit, c) at the same rate in every period. An LP model, such as that just considered, gives exactly the same recommendation as PV analysis if the same assumptions are made. LP analysis, however, allows one to have a borrowing rate different from a lending rate; a borrowing rate or lending rate that varies from period to period; a rate that depends upon the term of the loan (longer term usually means a higher rate/year); and/or an upper limit on the amount borrowed or lent at a given rate.

Like PV analysis, LP analysis avoids the ambiguity of multiple rates of return that can occur when the internal rate of return is used to evaluate a project. Consider a project that requires an initial investment of \$1 million, pays back \$2.5 million after one year, and incurs a termination cost after two years of \$1.55 million. This project has two internal rates of return. One is about 13.82% per year. The other is about 36.18% per year. Is the project attractive if our cost of capital is 11% per year? Both PV and LP analysis will (correctly) reject this project if our cost of capital is 12% per year, accept the project if our cost of capital is 24% per year, and reject the project if our cost of capital is 38% per year.

## 9.6 Accounting for Income Taxes

Suppose we take taxes into account. Let us consider the following simplified situation. There is a tax rate of fifty percent on profit for any period. If there is a loss in a period, eighty percent can be carried

forward to the next period. (Typically, tax laws put a limit on how many years a loss can be carried forward, but eighty percent may be a good approximation.)

Taxable income for each of the prospective projects as well as all existing projects is given in the table below. Note that because of factors such as depreciation, actual net cash flow may be rather different from taxable income in a period:

Period	Project			
	Foster City	Lower-Middle Housing	Disney Universe	Existing
1	-100,000	-200,000	-150,000	0
2	-300,000	-400,000	-200,000	100,000
3	-600,000	-200,000	-300,000	80,000
4	-100,000	500,000	-200,000	76,000
5	500,000	1,000,000	500,000	72,000
6	1,000,000	100,000	800,000	68,000
7	4,000,000	-1,000,000	5,000,000	60,000

To formulate a model, in this case, we need to additionally define:

$P_i$  = profit in period  $i$ , and

$C_i$  = loss in period  $i$ .

The formulation is affected in two ways. First, we must append some equations that force the  $P_i$ 's and  $C_i$ 's to be computed properly, and, secondly, terms must be added to the cash flow constraints to account for the cash expended in the payment of tax.

In words, one of the tax computation equations is:

$$\text{Profit} - \text{loss} = \text{revenue} - \text{expense} - 0.8 \times (\text{last period's loss}).$$

Algebraically, this equation for period 2 is:

$$P_2 - C_2 = 100 + 0.03L_1 - 300F - 400M - 200D - 0.035B_1 - 0.8C_1,$$

or in standard form:

$$P_2 - C_2 - 0.03L_1 + 300F + 400M + 200D + 0.035B_1 + 0.8C_1 = 100.$$

The entire formulation is:

```

MAX = Z;
!Cash flow constraints, uses-sources= 0, including the 50% tax usage;
3000*F +2000*M + 2000*D - B1 + L1 + 0.5*P1=2000;
1000*F + 500*M + 2000*D+1.035*B1-1.03*L1-B2+L2+0.5*P2= 500;
1800*F -1500*M + 1800*D+1.035*B2-1.03*L2-B3+L3+0.5*P3= 400;
-400*F -1500*M - 1000*D+1.035*B3-1.03*L3-B4+L4+0.5*P4= 380;
-1800*F -1500*M - 1000*D+1.035*B4-1.03*L4-B5+L5+0.5*P5= 360;
-1800*F - 200*M - 1000*D+1.035*B5-1.03*L5-B6+L6+0.5*P6= 340;
Z-5500*F+1000*M - 6000*D+1.035*B6-1.03*L6 +0.5*P7= 300;
! The borrowing limits;
B1 <= 2000;
B2 <= 2000;
B3 <= 2000;
B4 <= 2000;
B5 <= 2000;
B6 <= 2000;
! The investing limits;
F <= 1;
M <= 1;
D <= 1;
! Taxable Profit-Loss for each period;
100*F+ 200*M+ 150*D +P1 -C1 = 0;
300*F+ 400*M+ 200*D+0.035*B1-0.03*L1+P2+0.8*C1-C2=100;
600*F+ 200*M+ 300*D+0.035*B2-0.03*L2+P3+0.8*C2-C3= 80;
100*F- 500*M+ 200*D+0.035*B3-0.03*L3+P4+0.8*C3-C4= 76;
-500*F-1000*M- 500*D+0.035*B4-0.03*L4+P5+0.8*C4-C5= 72;
-1000*F- 100*M- 800*D+0.035*B5-0.03*L5+P6+0.8*C5-C6= 68;
-4000*F+1000*M-5000*D+0.035*B6-0.03*L6+P7+0.8*C6-C7= 60;

```

The solution is:

Variable	Value	Reduced Cost
Objective value:	5899.975	
Z	5899.9750	0.0000000
F	0.4872107	0.0000000
M	1.0000000	0.0000000
D	0.0000000	945.00740
B1	1461.6320	0.0000000
L1	0.0000000	0.5111823E-02
P1	0.0000000	0.4499472
B2	2000.0000	0.0000000
L2	0.0000000	0.1960928
P2	0.0000000	0.3793084
B3	1046.9790	0.0000000
L3	0.0000000	0.3167932E-02
P3	0.0000000	0.2042549
B4	0.0000000	0.2575563E-02
L4	991.26070	0.0000000
P4	0.0000000	0.1107492
B5	0.0000000	0.2537532E-02
L5	3221.6490	0.0000000
P5	1072.6580	0.0000000

B6	0.0000000	0.2499981E-02
L6	4359.3480	0.0000000
P6	751.86020	0.0000000
P7	1139.6230	0.0000000
C1	248.72110	0.0000000
C2	696.29720	0.0000000
C3	1039.3640	0.0000000
C4	340.85670	0.0000000
C5	0.0000000	0.1091125
C6	0.0000000	0.1075000
C7	0.0000000	0.5000000
Row	Slack or Surplus	Dual Price
1	5899.9750	1.0000000
2	0.0000000	1.3218740
3	0.0000000	1.2860920
4	0.0000000	1.0678540
5	0.0000000	1.0456780
6	0.0000000	1.0302250
7	0.0000000	1.0150000
8	0.0000000	1.0000000
9	538.36780	0.0000000
10	0.0000000	0.1924019
11	953.02070	0.0000000
12	2000.0000	0.0000000
13	2000.0000	0.0000000
14	2000.0000	0.0000000
15	0.5127893	0.0000000
16	0.0000000	573.56060
17	1.0000000	0.0000000
18	0.0000000	-0.2109901
19	0.0000000	-0.2637376
20	0.0000000	-0.3296720
21	0.0000000	-0.4120900
22	0.0000000	-0.5151125
23	0.0000000	-0.5075000
24	0.0000000	-0.5000000

Notice tax considerations cause a substantial change in the solution. More funds are placed into the lower-middle income housing project, *M*, and fewer funds are invested in the Foster City project, *F*. Project *M* has cash flows, which help to smooth out the stream of yearly profits.

## 9.7 Dynamic or Multi-period Networks

Thus far we have viewed network problems mainly as either a steady state or a one period problem. For example, in a pipeline network model, the solution can be interpreted as either the flow of material that occurs continuously day after day, or as a flow that occurs for one period and then stops. In many real systems, however, we are interested in a flow that varies from period to period, i.e., we are interested in multi-period or dynamic solutions. In these multi-period flows we also want to take into account that it may take several periods for flow to travel over an arc. Example dynamic networks are: a) river systems with various dams where we are interested in the amount of water to be spilled over the dam each period so as to satisfy various criteria regarding lake and river levels, river flows, and hydroelectric generation needs. A period might be a day, an arc might be the river section from one dam to the next, and it may take several periods for water to flow from one dam to the next. b) evacuation of a threatened facility or region as part of disaster planning, where we are interested in what routes people should take so that a large number of people escape in a short amount of time. For a building evacuation, a period might be 10 seconds, an arc may be a hallway, or a stairwell from one door to the next. Each arc may have a capacity limit of how many people can enter it per period, c) fleet routing of airplanes or trucks, where each arc in the network is a movement that must be made by either a truck or an airplane. The lead time of an arc is the length of time that it takes a vehicle to traverse the arc.

To represent a dynamic network algebraically, we need to define:

Parameters:

$L(i,j)$  = lead time, in periods, for flow to travel from node  $i$  to node  $j$  in the arc from  $i$  to  $j$ ,

Variables:

$x_{ijt}$  = flow entering arc  $ij$  at  $i$  in period  $t$ , and therefore exiting at  $j$  in period  $t+L(i,j)$ ,  
 $V_{jt}$  = inventory remaining at node  $j$  at the end of period  $t$ ,

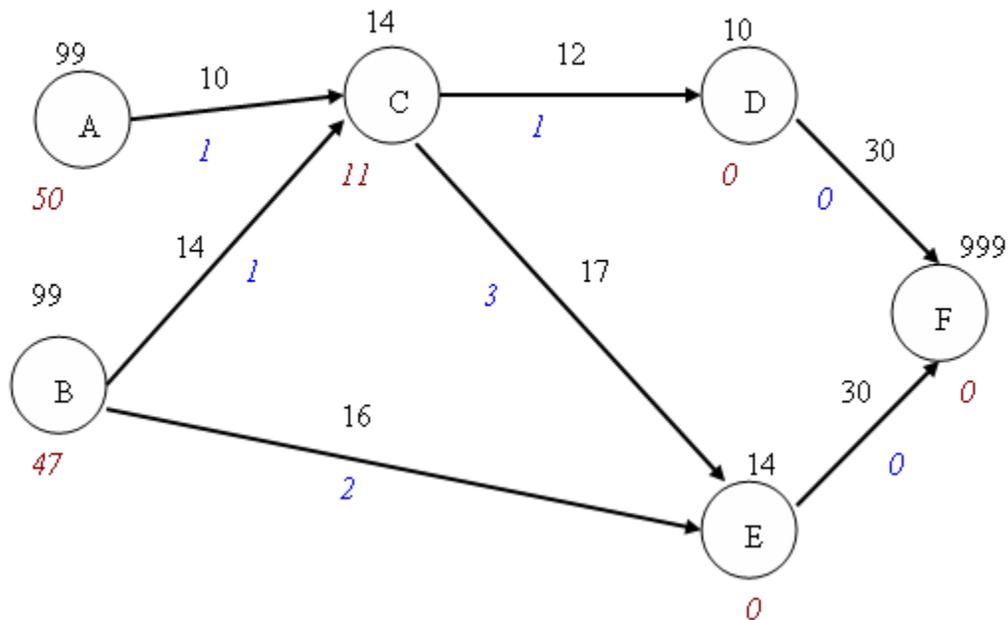
The basic node balance equation says that (inventory at node  $k$  at the end of period  $t$ ) = (ending inventory at  $k$  in the preceding period) + (arriving shipments) – (shipments out of  $k$  in  $t$ ), or algebraically:

$$V_{kt} = V_{kt-1} + \sum_i x_{ik(t-L(i,k))} - \sum_j x_{kjt}$$

Example:

We will illustrate the main ideas with an evacuation problem for a building. Complete details can be found in the set based LINGO model: `evacu8.lng` in the Applications Library at [www.lindo.com](http://www.lindo.com). Figure 9.1 gives the numeric details of a building for which we want to plan evacuation routes. The nodes are places where people are or can congregate. The arcs correspond to hallways, stairwells, etc. The number of people to be evacuated from each node in the network appears in italicized font below each node. A period is a 10 second interval. The italicized number appearing below each arc is the number of periods it takes to traverse the arc. The number appearing above each arc is the upper limit on the number of people that can enter an arc per period. The number appearing above each node is the upper limit on the number of people that can be waiting at a node.

Figure 9.1 Building Evacuation Network



Node F corresponds to the outside world. For example, the fastest that one of the 50 people at node A can get to the safety of the outside world is in 2 periods by taking the path A, C, D, F. Not all the people at A can make it this fast, however, because the arc from C to D can handle only 12 people per period. Also, people from B may also try to use arc C, D. Arc (C,E) with its long lead time of three periods but relatively high capacity of 17 might correspond to a long, wide corridor, whereas arc (A,C) with its short lead time but low capacity might correspond to a short narrow stairwell.

What should be our objective? An obvious one is to minimize the number of periods that it takes to evacuate all people. An interesting challenge is to see if you can do it in at most 70 seconds for the above problem. Perhaps a more refined objective is to maximize the number of people who get out quickly. If  $X_{i_j_t}$  is the number of people moving from  $i$  to  $j$  starting in period  $t$ , and realizing that node F is the outside world, we would like  $X_{D_F_1} + X_{E_F_1}$  to be large, and  $X_{D_F_9} + X_{E_F_9}$  to be much smaller. The objective that we will in fact use is:

$$\begin{aligned} \text{Max} = & 10*(X_{D_F_1} + X_{E_F_1}) + 9*(X_{D_F_2} + X_{E_F_2}) \\ & + 8*(X_{D_F_3} + X_{E_F_3}) + 7*(X_{D_F_4} + X_{E_F_4}) + \text{etc.} \end{aligned}$$

That is, we attach a desirable weight of 10 to getting people out in period 1, a weight of 9 to getting them out in period 2, etc. The model `evacu8.lng` is written in very general SETS form. If you want to see what the actual objective (as shown above) or constraints look like for the given data set, click on [LINGO | Generate | Display model](#).

Suppose you allow 10 periods for our model. We do not draw the corresponding multiperiod network, but you can think of drawing it as follows. Get yourself a very wide sheet of paper and make 10 copies, side by side of the above network. Then unhook the arrow end of each arc  $(i, j)$  and reconnect it  $L(i, j)$  subnetworks later. The main, nontrivial constraints in this network model are the flow balance constraints at each node each period. For example, the constraint for node E in period 5 is:

$$[\text{BAL\_E\_5}] - X_{B\_E\_3} - X_{C\_E\_2} + X_{E\_F\_5} - V_{E\_4} + V_{E\_5} = 0;$$

This is equivalent to:

$$V_{E\_5} = V_{E\_4} + X_{B\_E\_3} + X_{C\_E\_2} - X_{E\_F\_5};$$

In words, this says that at the end of period 5, the number of people at node E equals the number there at the end of period 4, plus people that left node B for E two periods ago, plus the number of people that left node C for E three periods ago, minus the number of people that left node E in period 5 for node F.

In general SETS form in `evacu8.lng`, this constraint is written as:

```
! For every node k and time period t;
@FOR( NXT( k, t) | t #GT# 1:
[BAL] V(k, t) = V(k, t-1) - @SUM( NXN( k, j) : X( k, j, t) )
      +@SUM( NXN( i, k) | t-LT( i, k) #GT# 0: X( i, k, t-LT( i, k) ) ) ;
);
```

where the set `NXT( , )` is the set of all node  $k$ , time period  $t$  combinations, and the set `NXN( , )` is the set of all from-to arcs  $k, j$  that exist in the network.

The model is completed by adding the upper bound constraints on the number of people at each node each period, and the upper bound constraints on the number of people traveling on each arc each period. For example, the flow upper bound on the arc from B to E in period 4 is:

$$[\text{UFLO\_B\_E\_4}] X_{B\_E\_4} \leq 16;$$

The upper bound on the number of people at node D at the end period 6 is:

$$[\text{USTOR\_D\_6}] V_{D\_6} \leq 10;$$

If you solve `evacu8.lng`, you will see that you can in fact evacuate the building in 70 seconds. For simplicity and ease of direction, e.g. in terms of placement of “Exit This Way” signs, it might be desirable that the solution have all people at a given node evacuate over the same route. You may wish to check whether the solution satisfies this additional “administrative” constraint. Another example of a dynamic network, this time for a hydroelectric river system can be found in the model `dampoold.lng`. For a production example, see `mrpcap.lng`.

## 9.8 End Effects

Most multi-period planning models “chop” off the analysis at some finite time in the future. The manner in which this chopping off is done can be important. In general, we care about the state in which things are left at the end of a planning model (e.g., inventory levels and capital investment). If we arbitrarily terminate our planning model at year five in the future, then an optimal solution to our model may, in reality, be an optimal solution to how to go out of business in five years. Grinold (1983) provides a comprehensive discussion of various methods for mitigating end-of-horizon effects. Some of the options for handling the end effect are:

- a) *Truncation*. Simply drop from the model all periods beyond a chosen cutoff point.
- b) *Primal limits*. Place reasonable limits on things such as inventory level at the end of the final period.
- c) *Salvage values/ dual prices*. Place reasonable salvage values on things such as inventory level at the end of the final period.
- d) *Infinite final period*. Let the final period of the model represent an infinite number of periods for which the same decisions are made in every period. So, for example, ending inventories = beginning inventories in this final period. Net present value discounting is used in the objective function to make the final period comparable to the earlier finite periods. This is the approach used by Carino et al. (1994) in their model of the Yasuda Kasai Company, Peiser and Andrus (1983) in their model of Texas real estate development, and by Eppen, Martin, and Schrage (1988) in their model of General Motors.

### 9.8.1 Perishability/Shelf Life Constraints

Many products, food products in particular, are perishable. It is important to take into account the fact the product can be stored in inventory for only a modest length of time. For example, blood for blood transfusions can be stored for at most 21 days. If there is a single level of production, then this consideration is easy to represent. Define:  $d_t$  = demand in period  $t$  (given), and the variables:  $P_t$  = production in period  $t$ , and  $I_t$  = inventory at the end of period  $t$ . Then, the standard inventory balance constraint is:

$$I_{t-1} + P_t = d_t + I_t$$

If product can be carried for one period before it is discarded, then it is clear that we should add the constraint:  $I_t \leq d_{t+1}$ . In general, if product can be carried in inventory for at most  $k$  periods, then we add the constraint:  $I_t \leq d_{t+1} + d_{t+2} + \dots + d_{t+k}$ .

### 9.8.2 Startup and Shutdown Costs

In the electric power generation industry, there is a decision problem known as the unit commitment problem. As the power demanded over the course of a day varies, the power generation company must decide which power units to start up as the demand increases and which to shutdown as demand decreases. A major concern is that there may be a significant cost to startup a generator, regardless of how long it runs. It is usually the case that the unit that is more efficient at producing power (e.g., a coal-fired unit) may, however, cost more to startup than say a gas-fired unit. Thus, if an extra burst of power is needed for only a short interval of time, it may be more cost effective to start up and run the gas-fired unit. A similar cost structure was encountered by Eppen, Martin, and Schrage (1988) in planning startup and shutdown of automotive plants. The typical way of representing startup costs, as well as shutdown costs, is with the following three sets of variables:  $y_{it} = 1$  if unit  $i$  is operating in period  $t$ , else 0;  $z_{it} = 1$  if unit  $i$  is started in period  $t$ , else 0;  $q_{it} = 1$  if unit  $i$  stops in period  $t$ , else 0.

The crucial constraints are then:

$$z_{it} - q_{it} = y_{it} - y_{it-1}.$$

Thus, if  $y_{it} = 1$ , but  $y_{it-1} = 0$ , then  $z_{it}$  is forced to be 1. If  $y_{it} = 0$ , but  $y_{it-1} = 1$ , then  $q_{it}$  is forced to be 1. For completeness, you may also need  $z_{it} + q_{it} \leq 1$ , and  $z_{it}, q_{it}$  restricted to 0 or 1.

## 9.9 Non-optimality of Cyclic Solutions to Cyclic Problems

In some situations, such as when modeling the end of the planning horizon as above, it is reasonable to assume demand is cyclic (e.g., it repeats forever in a weekly cycle). A natural question to ask is whether an optimal policy will have the same cycle length. We shall see that the answer may be 'no'. That is, even though demand has the same pattern, week-in and week-out, the most profitable policy need not have a weekly cycle. It may be optimal to behave differently from week to week.

In order to illustrate, let us reconsider the fleet routing and assignment problem introduced in chapter 8. We augment the original data with data on the profitability of two aircraft types for each flight:

Flight	Origin	Dest.	Depart	Arrive	Profit contribution (\$100)	
					MD90	B737
F221	ORD	DEN	800	934	115	111
F223	ORD	DEN	900	1039	109	128
F274	LAX	DEN	800	1116	129	104
F105	ORD	LAX	1100	1314	135	100
F228	DEN	ORD	1100	1423	125	102
F230	DEN	ORD	1200	1521	132	105
F259	ORD	LAX	1400	1609	112	129
F293	DEN	LAX	1400	1510	105	131
F412	LAX	ORD	1400	1959	103	135
F766	LAX	DEN	1600	1912	128	105
F238	DEN	ORD	1800	2121	128	101

For example, on flight pattern 221 an MD90 aircraft is more profitable than a B737 (\$11,500 vs. \$11,100), whereas a B737 is substantially more profitable (\$12,900 vs. \$11,200) on flight pattern 259. The above pattern of flights is to be covered every day. Suppose that we have seven MD90's available, but only one B737 available to cover these flights. As before, we assume no deadheading. First, we assume that we will use a solution with a cycle of one day. An appropriately modified model from chapter 8 is:

```

MODEL:
SETS: ! Fleet routing and assignment (FLEETRAT);
CITY ;; ! The cities involved;
ACRFT: ! Aircraft types;
FCOST, ! Fixed cost per day of this type;
FSIZE; ! Max fleet size of this type;
FLIGHT;;
FXCXC( FLIGHT, CITY, CITY) :
DEPAT, ! Flight departure time;
ARVAT; ! arrival time at dest.;
AXC( ACRFT, CITY) :
OVNITE; ! Number staying overnight by type, city;
AXF( ACRFT, FXCXC) :
X, ! Number aircraft used by type, flight;
PC; ! Profit contribution by type, flight;
ENDSETS
DATA:
CITY = ORD DEN LAX;
ACRFT, FCOST, FSIZE =
MD90 .01 7
B737 .01 1;
FLIGHT = F221 F223 F274 F105 F228 F230 F259 F293 F412 F766 F238;

```

```

FXCXC, DEPAT, ARVAT =
!   Flight  Origin  Dest.  Depart  Arrive;
      F221  ORD    DEN    800    934
      F223  ORD    DEN    900    1039
      F274  LAX    DEN    800    1116
      F105  ORD    LAX    1100   1314
      F228  DEN    ORD    1100   1423
      F230  DEN    ORD    1200   1521
      F259  ORD    LAX    1400   1609
      F293  DEN    LAX    1400   1510
      F412  LAX    ORD    1400   1959
      F766  LAX    DEN    1600   1912
      F238  DEN    ORD    1800   2121;

PC =   ! Profit contribution of each vehicle*flight combo;
      115     109     129     135     125     132
      112     105     103     128     128
      111     128     104     100     102     105
      129     131     135     105     101;

ENDDATA
!-----;
! Maximize profit contribution from flights minus
  overhead cost of aircraft in fleet;
MAX = @SUM( AXF( I, N, J, K): PC( I, N, J, K) * X( I, N, J, K) )
      - @SUM( AXC( I, J): FCOST( I) * OVNITE( I, J) );
! At any instant, departures in particular, the number of
  cumulative arrivals must be >= number of cumulative departures;
! For each flight of each aircraft type;
@FOR( ACFT( I):
  @FOR( FXCXC( N, J, K):
! Aircraft on ground in morning +
  number aircraft arrived thus far >=
  number aircraft departed thus far;
  OVNITE( I, J) +
  @SUM( FXCXC( N1, J1, K1) | K1 #EQ# J #AND#
        ARVAT( N1, J1, K1) #LT# DEPAT( N, J, K):
        X( I, N1, J1, J) ) >=
  @SUM( FXCXC( N1, J1, K1) | J1 #EQ# J #AND#
        DEPAT( N1, J1, K1) #LE# DEPAT( N, J, K):
        X( I, N1, J, K1) );
  ););
! This model does not allow deadheading, so at the end of the day,
  arrivals must equal departures;
@FOR( ACFT( I):
  @FOR( CITY( J):
    @SUM( AXF( I, N, J1, J): X( I, N, J1, J) ) =
    @SUM( AXF( I, N, J, K): X( I, N, J, K) );
  );
);
! Each flight must be covered;
@FOR( FXCXC( N, J, K):
  @SUM( AXF( I, N, J, K): X( I, N, J, K) ) = 1;
);

```

```

! Fleet size limits;
  @FOR( ACRFT( I):
    @SUM( AXC( I, J): OVNITE( I, J) <= FSIZE( I);
  );
! Fractional planes are not allowed;
  @FOR( AXF: @GIN( X); );
END

```

It has the solution:

```

Global optimal solution found at step:          106
Objective value:                             1323.940

```

Variable	Value
X( MD90, F221, ORD, DEN)	1.000000
X( MD90, F223, ORD, DEN)	1.000000
X( MD90, F274, LAX, DEN)	1.000000
X( MD90, F105, ORD, LAX)	1.000000
X( MD90, F228, DEN, ORD)	1.000000
X( MD90, F230, DEN, ORD)	1.000000
X( MD90, F259, ORD, LAX)	1.000000
X( MD90, F412, LAX, ORD)	1.000000
X( MD90, F238, DEN, ORD)	1.000000
X( B737, F293, DEN, LAX)	1.000000
X( B737, F766, LAX, DEN)	1.000000

The daily profit contribution of this solution is  $1323.94 * 100 = \$132,394$  per day. Notice that our single B737 flies from *DEN* at 2 pm to *LAX* as flight 293, and then departs *LAX* at 4 pm for *DEN* as flight 766. The above model requires that at the beginning of each day we must have the same number of MD90's and B737's at a given airport as on every other day. Just for reference, if you solve the above model with no B737's available, the profit contribution is \$132,094. So, the B737 seems to be worth only \$200 per day.

Can we do better if we allow a two-day cycle in the solution? We can try by changing the input to the model as in the model below. Effectively, we have given two days worth of demand, denoting the second day's flights by an *S*, vs. the *F* denoting the flights on the first day. Otherwise, the model is identical. The profit of this two day solution should be at least  $2 * 132,394 = \$264,788$ :

```

MODEL:
SETS: ! Fleet routing and assignment (FLEETRAT);
  CITY ;; ! The cities involved;
  ACRFT: ! Aircraft types;
  FCOST, ! Fixed cost per day of this type;
  FSIZE; ! Max fleet size of this type;
  FLIGHT;;
  FXCXC( FLIGHT, CITY, CITY) :
  DEPAT, ! Flight departure time;
  ARVAT; ! arrival time at dest.;
  AXC( ACRFT, CITY):
  OVNITE; ! Number staying overnight by type, city;
  AXF( ACRFT, FXCXC):
  X, ! Number aircraft used by type, flight;
  PC; ! Profit contribution by type, flight;
ENDSETS

```

## DATA:

```

CITY = ORD DEN LAX;
ACRFT, FCOST, FSIZE =
  MD90 .01 7
  B737 .01 1;
FLIGHT = F221 F223 F274 F105 F228 F230 F259 F293 F412 F766 F238
          S221 S223 S274 S105 S228 S230 S259 S293 S412 S766 S238;
FXCXC, DEPAT, ARVAT =
! Flight Origin Dest. Depart Arrive;
  F221 ORD DEN 800 934
  F223 ORD DEN 900 1039
  F274 LAX DEN 800 1116
  F105 ORD LAX 1100 1314
  F228 DEN ORD 1100 1423
  F230 DEN ORD 1200 1521
  F259 ORD LAX 1400 1609
  F293 DEN LAX 1400 1510
  F412 LAX ORD 1400 1959
  F766 LAX DEN 1600 1912
  F238 DEN ORD 1800 2121
  S221 ORD DEN 3200 3334
  S223 ORD DEN 3300 3439
  S274 LAX DEN 3200 3516
  S105 ORD LAX 3500 3714
  S228 DEN ORD 3500 3823
  S230 DEN ORD 3600 3921
  S259 ORD LAX 3800 4009
  S293 DEN LAX 3800 3910
  S412 LAX ORD 3800 4359
  S766 LAX DEN 4000 4312
  S238 DEN ORD 4000 4521;
PC = ! Profit contribution of each vehicle*flight combo;
  115 109 129 135 125 132
  112 105 103 128 128
  115 109 129 135 125 132
  112 105 103 128 128
  111 128 104 100 102 105
  129 131 135 105 101
  111 128 104 100 102 105
  129 131 135 105 101;

```

ENDDATA

Now, the solution is:

Global optimal solution found at step: 103  
 Objective value: 2718.930

Variable	Value
X( MD90, F221, ORD, DEN)	1.000000
X( MD90, F223, ORD, DEN)	1.000000
X( MD90, F274, LAX, DEN)	1.000000
X( MD90, F105, ORD, LAX)	1.000000
X( MD90, F228, DEN, ORD)	1.000000
X( MD90, F230, DEN, ORD)	1.000000
X( MD90, F259, ORD, LAX)	1.000000
X( MD90, F293, DEN, LAX)	1.000000
X( MD90, F766, LAX, DEN)	1.000000
X( MD90, F238, DEN, ORD)	1.000000
X( MD90, S221, ORD, DEN)	1.000000
X( MD90, S274, LAX, DEN)	1.000000
X( MD90, S105, ORD, LAX)	1.000000
X( MD90, S228, DEN, ORD)	1.000000
X( MD90, S230, DEN, ORD)	1.000000
X( MD90, S259, ORD, LAX)	1.000000
X( MD90, S412, LAX, ORD)	1.000000
X( MD90, S766, LAX, DEN)	1.000000
X( MD90, S238, DEN, ORD)	1.000000
X( B737, F412, LAX, ORD)	1.000000
X( B737, S223, ORD, DEN)	1.000000
X( B737, S293, DEN, LAX)	1.000000

Notice that our profit,  $2718.93 * 100 = \$271,893$  is more than twice the profit of the one day solution,  $2 * 132,394 = \$264,788$ . How did we arrive at this happy situation? Notice how the B737 is used. On the first day, it flies from *LAX* to *ORD* via flight 412. On the second day, it flies from *ORD* to *DEN* via flight 223 and then from *DEN* back to *LAX* via flight 293. It is only on the second day that it is back where it started, *LAX*. All three flights are very profitable for the B737 relative to the MD90. By allowing a two-day cycle, the B737 is able to cover these very profitable flights at least half of the time. Thus, even though the demand pattern has a one day cycle, it is profitable to allow the solution to have a two day cycle.

A good discussion of how to avoid the temptation to restrict solutions can be found in the book on “conceptual blockbusting” by Adams (1986). Orlin (1982) gives a more detailed analysis of the cyclic vehicle routing problem.

## 9.10 Problems

1. The Izza Steel Company of Tokyo has predicted delivery requirements of 3,000, 6,000, 5,000, and 2,000 tons of steel in the next four periods. Current workforce is at the 4,000 tons per period level. At the moment, there is 500 tons of steel in stock. At the end of the four periods, Izza would like its inventory position to be back at 500 tons. Regular time workforce has a variable cost of \$100 per ton. Overtime can be hired in any period at a cost of \$140 per ton. Regular time workforce size can be increased from one period to the next at a cost of \$300 per ton of change in capacity. It can be decreased at a cost of \$80 per ton. There is a charge of \$5 per ton for inventory at the end of each period. Izza would like the regular time workforce to be at the 3,000-ton level at the end of the four periods.
  - a) Formulate Izza's problem as a linear program.
  - b) What assumption does your model make about idle workforce?
2. An airline predicts the following pilot requirements for the next five quarters: 80, 90, 110, 120, 110. Current staff is 90 pilots. The question of major concern is the number of pilots to hire in each of the next five quarters. A pilot must spend the quarter in which she is hired in training. The line's training facilities limit the number of pilots in training to at most 15. Further, the training of pilots requires the services of experienced pilots at the ratio of 5 to 1 (e.g., five pilots in training require one experienced pilot). An experienced pilot so assigned cannot be used to satisfy regular requirements. The cost of hiring and training a pilot is estimated at \$20,000 exclusive of the experienced pilot time required. Experienced pilots cost \$25,000 per quarter. Company policy does not include firing pilots.
  - a) What are the variables?
  - b) Formulate a model for determining how many pilots to hire in each period.
3. The Touted de Suite Candy Company includes in its product line a number of different mixed nut products. The Chalet nut mix is required to have no more than 25 percent peanuts and no less than 40 percent almonds.

The nuts available, their prices, and their availabilities this month are as follows:

<b>Nut</b>	<b>Price</b>	<b>Availability</b>
Peanuts	20¢/lb.	400 lbs.
Walnuts	35¢/lb.	No limit
Almonds	50¢/lb.	200 lbs.

The Chalet mix sells for 80 cents per pound. At most, 700 pounds can be mixed per month in questions (a), (b), and (c).

- a) Formulate the appropriate model for this problem.
- b) Touted de Suite would like to incorporate into the analysis its second major mixed nut line, the Hovel line. The Hovel mix can contain no more than 60 percent peanuts and no less than 20 percent almonds. Hovel sells for 40 cents per pound. Modify your model appropriately.

- c) Toute de Suite would like to incorporate next month's requirements into the analysis. The expected situation next month is:

Nut	Price	Requirement (Availability)
<i>Peanuts</i>	19¢/lb.	500 lbs
<i>Walnuts</i>	36¢/lb.	No limit
<i>Almonds</i>	52¢/lb.	180 lbs.
<i>Chalet</i>	81¢/lb.	
<i>Hovel</i>	39¢/lb.	

It cost 2 cents per pound to store nuts (plain or mixed) for one month. Because of a contract commitment, at least 200 pounds of Chalet mix must be sold next month. Modify your model appropriately.

- If two parties to a financial agreement,  $A$  and  $B$ , want the agreement to be treated as a lease for tax purposes, the payment schedule typically must satisfy certain conditions specified by the taxing agency. Suppose  $P_i$  is the payment  $A$  is scheduled to make to  $B$  in year  $i$  of a seven-year agreement. Parties  $A$  and  $B$  want to choose at the outset a set of  $P_i$ 's to satisfy a tax regulation that no payment in any given period can be less than two-thirds of the payment in any later period. Show the constraints for period  $i$  to enforce this lower bound on  $P_i$ . Use as few constraints per period as possible.
- One of the options available to a natural gas utility is the renting of a storage facility, so it can buy gas at a cheap rate in the summer and store it until possibly needed in the winter. There is a yearly fee of \$80,000 for each year the facility is rented. There is an additional requirement that, if the utility starts renting the facility in year  $t$ , it must also rent it for at least the next three years. The gas utility has a long range planning model with a variable  $x_t = 1$  if the utility rents the storage facility in year  $t$ , 0 otherwise;  $y_t = 1$  if the utility starts renting in period  $t$ ; and  $z_t = 1$  if the utility stops renting after period  $t$ , for  $t = 1$  to 25. It is not clear whether or not this facility should be rented. Show how to represent this fee structure in an LP/IP model.
- Below is the formulation of a cash flow matching problem, where the  $B$  variables represent investments in bonds and the  $S$  variables represent investment in savings for one period. The right-hand sides are the cash requirements for the various years.

```

MIN = L;
[P0]L -.98 * B1 - .965 * B2 - S0 = 10;
[P01] .06 * B1 + .065 * B2 + 1.04 * S0 - S1 = 11;
[P02] .06 * B1 + .065 * B2 + 1.04 * S1 - S2 = 12;
[P03] .06 * B1 + .065 * B2 + 1.04 * S2 - S3 = 14;
[P04] .06 * B1 + .065 * B2 + 1.04 * S3 - S4 = 15;
[P05] 1.06 * B1 + .065 * B2 + 1.04 * S4 - S5 = 17;
[P06] .065 * B2 + 1.04 * S5 - S6 = 19;
[P07] .065 * B2 + 1.04 * S6 - S7 = 20;
[P08] .065 * B2 + 1.04 * S7 - S8 = 22;
[P09] .065 * B2 + 1.04 * S8 - S9 = 24;
[P10] .065 * B2 + 1.04 * S9 - S10 = 26;
[P11] .065 * B2 + 1.04 * S10 - S11 = 29;
[P12] 1.065 * B2 + 1.04 * S11 - S12 = 31;
END

```

- a) The option to borrow at seven percent per period for a term of one period has become available in every period. Show the modification in the above model for the periods with right-hand sides of 15, 17, and 19. Denote the borrowing variables by  $M_0$ ,  $M_1$ , etc.
- b) Would this option in fact be attractive in the above model in any period?
- c) Almost all the parties involved were happy with this model until the Internal Revenue Service (IRS) suddenly became interested. The IRS has made the judgment that the initial endowment in the very first period may be tax-free. However, thereafter, the regular tax laws apply. Upon further inquiry, the IRS responded that regular income is taxed at 37 percent and capital gains at 15 percent.

We now want to find the initial lump sum such that, after taxes have been paid each period, we can still cover the right-hand side requirements. For simplicity, assume taxes are paid in the same period as the income being taxed. Show how rows  $P_04$  and  $P_05$  are altered by this unpleasant new reality (Disregard (a) and (b) above in answering.).